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On Sextic Surfaces Having a Nodal Curve of Order 8.

By C. H. Sisam.

1. In an article in this Journal (Vol. XXXVII, p. 445), I have discussed sextic surfaces whose plane sections are of genus 1. In this paper, non-ruled* sextics whose plane sections are of genus 2 are considered.

2. It is known† that such a surface is rational and that it is generated by a pencil (faisceau) of conics. The developable of the planes of these conics is of class 3. For, the section of the surface by a generic plane π is an hyperelliptic sextic curve. The section by π of the system of planes of the generating conics is the system of lines joining corresponding points of the g_2' on the sextic. The involutorial (6, 6) correspondence of the lines of any pencil in π such that corresponding lines pass through pairs of points of g_2' has twelve coincidences. Six of these arise from coincidences of g_2'. The other six (in pairs) are due to lines in the pencil which join corresponding points of g_2'.

3. The system of conics on the given sextic surface is thus the locus of the conic of intersection of corresponding surfaces of the systems‡

$$L_1k_1^3+L_2k_1^2k_2+L_3k_1k_2^2+L_4k_2^3=0, \tag{1}$$

$$L_1L_5k_1^2+Qk_1k_2+L_4L_6k_2^2=0, \tag{2}$$

where $L_i=0$ is the equation of a plane, $Q=0$ is the equation of a quadric and k_1, k_2 are parameters.

4. *Six Conics of the System are Composite.* The condition that a conic of the system is composite is that its plane (1) touches the corresponding quadric (2). If we write the equation of a quadric (2) in plane coordinates and impose the condition that the corresponding plane (1) satisfies the resulting equation, we obtain an equation of degree 12 in k_1 and k_2. It has an

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* For the discussion of ruled sextics whose plane sections are of genus 2, see Snyder, in this Journal, Vol. XXVII, pp. 77-102, 173-188, where other references will be found.

† Castelnuovo-Enriques, *Math. Annalen,* Vol. XLVIII, p. 308.

‡ Compare an article by the author in this Journal, Vol. XXX, p. 99.

extraneous factor $k_1^3k_2^3$. The six values of k_1/k_2 which make the residual factor zero determine the composite conics of the system.

5. The given system of conics is also defined by (1) and any one of the systems of surfaces

$$(L_2L_5-Q)k_1^2+(L_3L_5-L_4L_6)k_1k_2+L_4L_5k_2^2=0, \tag{3}$$

$$L_1L_6k_1^2+(L_2L_6-L_1L_5)k_1k_2+(L_3L_6-Q)k_2^2=0, \tag{4}$$

$$(QL_6+L_1L_5^2-L_2L_5L_6)k_1+(QL_5+L_4L_6^2-L_3L_5L_6)k_2=0. \tag{5}$$

If the coordinates of a point on the surface are substituted in (2), (3) and (4), the resulting equations in k_1/k_2 have a common root. By eliminating k_1 and k_2 from these equations we obtain at once the equation of the surface:

$$\begin{vmatrix} L_1L_5 & Q & L_4L_6 \\ L_2L_5-Q & L_3L_5-L_4L_6 & L_4L_5 \\ L_1L_6 & L_2L_6-L_1L_5 & L_3L_6-Q \end{vmatrix}=0. \tag{6}$$

6. For points on the double curve, equations (2), (3) and (4) have both roots equal. Hence, the double curve lies on all the quartic surfaces determined by equating to zero the first minors of (6).

7. At a triple point of the surface, equations (2), (3) and (4) are satisfied identically. There are two such points, defined by

$$L_5=L_6=Q=0. \tag{7}$$

8. For points on the double curve, (5) is satisfied identically; so that the double curve of (6) lies on all the cubic surfaces of the pencil (5). The triple points (7) are double points on these surfaces. The residual basis curve of the pencil is the line $L_5=0L_6=0$ joining the triple points. The octavic component of the basis curve of (5) has the points (7) as triple points. It is of genus 3 since* its projection on a plane from a triple point is a quintic curve with a triple point and no double points.

9. The complete intersection of a cubic of the pencil (5) with the given surface (6) consists of the double octic and a generating conic. It follows at once that if C_n is a curve of order n on the given surface, α the number of its intersections with the double curve and β the number of its intersections with a generic generating conic, then

$$\alpha+\beta=3n,$$

since $\alpha+\beta$ is the number of its intersections with a generic cubic surface (5).

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* This result also follows from the vanishing of the numerical genus. Cf. Castelnuovo-Enriques, *loc. cit.*, p. 312.

In particular, the right lines on the surface (Art. 4) are trisecants of the double curve. The generating conics intersect it in six points.

10. Denote the equation (6) of the given surface by $\Phi=0$ and those of two cubics of the system (5) by $\phi_1=0$ and $\phi_2=0$. The surfaces of the linear system ∞^3

$$\lambda_1\Phi+\lambda_2\phi_1^2+\lambda_3\phi_1\phi_2+\lambda_4\phi_2^2=0 \tag{8}$$

all have the double curve of $\Phi=0$ as double curve. Every sextic surface on which this curve is double belongs to the system (8). For, its residual intersection with $F=0$ breaks up into two conics of the system (1), (2) since the conic of the system through a generic point of the residual intersection has thirteen points in common with each surface (Art. 9). The surface in question thus belongs to the pencil of sextics defined by $\Phi=0$ and a pair of cubics (5) and hence to the system (8).

It follows that, through a generic point of space, there passes a single conic which intersects the double octavic in six points. Every such conic lies on a cubic of the pencil (5). Its plane intersects the cubic on which it lies in a right line which intersects the line $L_5=0 L_6=0$ joining the triple points. The locus of this residual line is the ruled quartic

$$L_1L_5^3-L_2L_5^2L_6+L_3L_5L_6^2-L_4L_6^3=0.$$

Every plane tangent to this ruled quartic contains a conic of this doubly infinite system of conics.

The locus of the composite conics (Art. 4) on the system of surfaces (8) is the scroll of trisecants of the double octavic. This ruled surface has the octavic as sixfold curve. Its complete intersection with a surface (8) is the octavic and the six composite conics on the surface. It is of order 18.

11. Since the double octavic has two triple points and is of genus 3, the quartic surfaces which contain it form a linear system ∞^8 (cf. Art. 6). The residual octavic of intersection of such a quartic with $\Phi=0$ intersects the generating conics in two points (Art. 9), and the double octavic in twenty-two points. It is of genus 3.

12. The quartic surfaces of the system of Art. 11 which contain one component of each of three composite generating conics of $F=0$ form a bundle of quartics since the three lines are trisecants of the double octavic. The residual intersection of a quartic of the bundle with $\Phi=0$ is a quintic which intersects the generating conics in two points and the double octavic in thirteen points. It intersects each of the basis right lines of the bundle in two points.

For, a generic plane through such a basis line intersects $\Phi=0$ in a quintic and the given quartic surface in a cubic. Ten intersections of this quintic and cubic lie on the double octavic and one on each of the other two basis lines of the bundle so that only three are points of intersection of the given space quintic with the given plane.

The given quintic thus intersects any other quartic of the bundle in a single point not on the basis curve. Hence, any two quartics of the bundle determine uniquely a point on the surface. Moreover, a generic point on the surface is uniquely determined in this way, so that a system of parametric equations of the surface are determined by solving the equation $\Phi=0$ as simultaneous with

$$\frac{\psi_1}{\xi'_1}=\frac{\psi_2}{\xi'_2}=\frac{\psi_3}{\xi'_3}, \tag{9}$$

where $\psi_1=0$, $\psi_2=0$, $\psi_3=0$ are three suitably chosen quartics of the bundle.

13. Let

$$x_i=f'_i(\xi'_1,\ \xi'_2,\ \xi'_3) \qquad i=1,\ 2,\ 3,\ 4 \tag{10}$$

be the parametric equations of the given surface obtained in this way. The curves $\Sigma u_i f'_i(S')=0$ which correspond to the plane sections of $\Phi=0$ are quintics (since to a generic line in the parametric plane corresponds a quintic on $\Phi=0$) of genus equal to that of the corresponding plane sections (Art. 1). By a suitable birational transformation of the plane, they may be transformed into a system of quartic curves having a common double point.

14. Let such a birational transformation be effected and let

$$x_i=f_i(\xi_1,\ \xi_2,\ \xi_3) \qquad i=1,\ 2,\ 3,\ 4 \tag{11}$$

be the resulting parametric equations of the surface. Denote the fundamental double point of the quartics $\Sigma u_i f_i(\xi)=0$ by P_0. Since a right line has just six points in common with the surface, these quartics have $4^2-2^2-6=6$ simple points in common. Denote these points by P_1, P_2, ..., P_6, respectively.

Let $P_0P_1P_2$ be chosen as vertices of the triangle of reference in the ξ-plane so that

$$\begin{aligned} x_i=f_i(\xi)\equiv a_{0i}\xi_1^2\xi_3^2+a_{1i}\xi_1\xi_2\xi_3^2+a_{2i}\xi_2^2\xi_3^2+a_{3i}\xi_1^3\xi_3+a_{4i}\xi_1^2\xi_2\xi_3+a_{5i}\xi_1\xi_2^2\xi_3 \\ +a_{6i}\xi_2^3\xi_3+a_{7i}\xi_1^3\xi_2+a_{8i}\xi_1^2\xi_2^2+a_{9i}\xi_1\xi_2^3, \qquad i=1,\ 2,\ 3,\ 4 \end{aligned} \tag{12}$$

where the coefficients are chosen so that $P_3P_4P_5P_6$ also lie on the curves $\Sigma u_i f_i=0$.

15. To the pencil of lines through P_0 corresponds the system of conics which generate the surface. Thus, to the line $k_2\xi_1 = k_1\xi_2$ corresponds the conic

$$x_i = (a_{0i}k_1^2 + a_{1i}k_1k_2 + a_{2i}k_2^2)\,\xi_3^2 + (a_{3i}k_1^3 + a_{4i}k_1^2k_2 + a_{5i}k_1k_2^2 + a_{6i}k_2^3)\,\xi_3\xi_2$$
$$+ (a_{7i}k_1^3k_2 + a_{8i}k_1^2k_2^2 + a_{9i}k_1k_2^3)\,\xi_2^2. \qquad i = 1, 2, 3, 4 \qquad (13)$$

The six composite conics (Art. 4) of the system (13) are determined by the lines joining P_0 to $P_i (i = 1, 2, \ldots, 6)$. One component of such a conic corresponds to the line P_0P_i, the other, to the fundamental point* P_i. There are, in general (cf. Art. 34), no other right lines on the surface.

16. There are, on the surface, thirty-two conics which do not belong to the system (13). One of these corresponds to P_0 (cf. Clebsch, *loc. cit.*) in such a way that to each direction through P_0 corresponds a point on the conic. Fifteen correspond to the lines joining the simple fundamental points in pairs; fifteen, to the conics through P_0 and four simple fundamental points. One other corresponds to the cubic which has a node at P_0 and passes through $P_1, P_2, \ldots, P_6$.

Each conic of the generating system (13) intersects each directrix conic in a single point. In particular, just one component of each composite conic intersects a given directrix conic. Denote the components of one of these composite conics by the symbols 1 and 1′, of another by 2 and 2′, etc., in such a way that the six lines 1, 2, 3, 4, 5, 6 all intersect the conic corresponding to P_0. If we put, in all possible ways, primes on an even number of the symbols 1, 2, 3, 4, 5, 6, we determine $2^5—1$ other sets of six lines on the surface such that all the lines of the set intersect a directrix conic. Let each such set of six symbols be used to indicate the directrix conic which intersects the corresponding lines. These thirty-two symbols individualize the thirty-two directrix conics.

Two directrix conics whose symbols have $2k$ corresponding component symbols unlike intersect in $k—1$ points as may be seen by counting the intersections, not at fundamental points, of the corresponding curves in the ξ-plane. Each directrix conic is intersected twice by just one other such conic, the one whose component symbols are all different from those of the given conic. Of two such pairs of conics, as, for example, 1 2 3 4 5 6, 1′ 2′ 3′ 4′ 5′ 6′ and 1′ 2′ 3 4 5 6, 1 2 3′ 4′ 5′ 6′, each conic of each pair intersects a single conic of the other pair. Given two non-intersecting conics, there are eight other conics which do not intersect either of them. Six directrix conics can be chosen (in thirty-two ways) so that no two intersect. Any five of six such conics intersect a right line lying on the surface.

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* Clebsch, *Math. Annalen*, Vol. I, p. 266.

Four directrix conics can be chosen in such a way that each intersects all the others just once. There are $\frac{32\cdot 15\cdot 6\cdot 1}{1\cdot 2\cdot 3\cdot 4}=120$ tetrads of this type.

17. The surface is generated by thirty-two linear systems of cubic curves. These systems of curves correspond to the six pencils of lines with vertices at $P_i (i=1, 2, \ldots, 6)$, the twenty pencils of conics through P_0 and three P_i, the six pencils of cubics which have a double point at P_0 and pass through five P_i. These thirty-two systems are characterized, in the notation of the preceding article, by putting primes on an odd number of the symbols, 1, 2, 3, 4, 5, 6. Two cubics which belong to systems having $2k$ component symbols unlike have k points in common. In particular, cubics belonging to the same system $(k=0)$ do not intersect. Each directrix conic intersects the cubics of any system which has $2k+1$ symbols unlike the symbol of the conic in k points. The generating conics intersect the cubics of each system just once.

There are twelve cubics on the surface each of which intersects each generating conic in two points. Six of these correspond to the six conics determined by five fundamental points $P_1, P_2, \ldots, P_6$. The others correspond to the cubics which pass through P_0, have a node at one of the points $P_1, \ldots, P_6$ and pass simply through the other five points. Each of these twelve cubics intersects one right line on the surface twice, the line coplanar with it not at all, and each of the remaining right lines on the surface once.

18. There are thirty-two linear systems each ∞^2 of rational quartics on the surface. One such system corresponds to the right lines in the ξ-plane, the others to the systems of curves which can be transformed into the right lines by birational transformations that transform the curves $\Sigma u_i f_i=0$ (Art. 13) into quartics. All the quartics of each system intersect six lines on the surface which intersect a directrix conic (Art. 16).

There are sixty pencils of rational quartics which intersect the generating conics twice. Any two non-intersecting lines on the surface determine such a system in such a way that the quartics of the system intersect each of the given lines twice.

There is a single linear system ∞^2 of quartics of genus 1 on the surface. It is determined by the system of cubics which pass through $P_0, P_1, \ldots, P_6$.

19. *The genus of a non-multiple curve of order n on the given surface can not exceed the greatest integer in* $\frac{(n-2)^2+9}{12}$. *This limit is attained for all values of n.*

Let C_n be the given curve of order n and let C_m, the corresponding curve in the ξ-plane, be of order m and have a point of multiplicity α_i at the fundamental point $P_i (i=0, 1, \ldots, 6)$. Then

$$4m-2\alpha_0-(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)=n,$$

since to each intersection of C_n with a generic plane $\Sigma u_i x_i=0$ corresponds an intersection, not at a fundamental point of C_m with $\Sigma u_i f_i=0$.

The curves C_m and C_n are in (1, 1) correspondence. Hence, the genus of C_n is not greater than

$$\frac{(m-1)(m-2)}{2}-\sum_{i=0}^{6}\frac{\alpha_i(\alpha_i-1)}{2}.$$

This function has its greatest value when

$$\alpha_0=\frac{2n-1}{6}\quad \alpha_1=\alpha_2=\alpha_3=\alpha_4=\alpha_5=\alpha_6=\frac{n+1}{6}\text{ whence } m=\frac{4n+1}{6}.$$

The value of the function for these values of the variables is $\frac{(n-2)^2+9}{12}$ so that the genus can not exceed the greatest integer in this expression. By giving suitable integer values to $\alpha_0, \alpha_1, \ldots, \alpha_6$, this limit can be attained for all values of n.

20. Six generating conics (13) touch a given plane (u). These correspond to the tangents from P_0 to the curve $\Sigma u_i f_i=0$ which corresponds to the section of the surface by the given plane. If we let

$$\alpha_i=u_1 a_{i1}+u_2 a_{i2}+u_3 a_{i3}+u_4 a_{i4}$$

we find, as the equation of these six tangents:

$$(\alpha_3\xi_1^2+\alpha_4\xi_1^2\xi_2+\alpha_5\xi_1\xi_2^2+\alpha_6\xi_2^3)^2-4\xi_1\xi_2(\alpha_7\xi_1^2+\alpha_8\xi_1\xi_2+\alpha_9\xi_2^2)$$
$$(\alpha_0\xi_1^2+\alpha_1\xi_1\xi_2+\alpha_2\xi_2^2)=0. \qquad (14)$$

The moduli of the section of the surface by the given plane are those of the corresponding quartic, that is, are the absolute invariants of the binary sextic (14). In particular, the condition that the given plane is tangent is that (14) has a double root. The equation of the surface in plane coordinates is thus found by equating to zero the discriminant of (14). The surface is of class 20.

21. The tangent plane at the point on the surface having the parameters

(ξ_1, ξ_2, ξ_3) determines a quartic in the ξ-plane with a double point at (ξ). The equation of this tangent plane is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ \frac{\partial f_1}{\partial \xi_1} & \frac{\partial f_2}{\partial \xi_1} & \frac{\partial f_3}{\partial \xi_1} & \frac{\partial f_4}{\partial \xi_1} \\ \frac{\partial f_1}{\partial \xi_2} & \frac{\partial f_2}{\partial \xi_2} & \frac{\partial f_3}{\partial \xi_2} & \frac{\partial f_4}{\partial \xi_2} \\ \frac{\partial f_1}{\partial \xi_3} & \frac{\partial f_2}{\partial \xi_3} & \frac{\partial f_3}{\partial \xi_3} & \frac{\partial f_4}{\partial \xi_3} \end{vmatrix} = 0. \tag{15}$$

Hence, the parametric equations of the surface in plane coordinates are

$$u_i = J_i(\xi_1, \xi_2, \xi_3), \tag{16}$$

where J_i is the cofactor of x_i in (15).

The curves $\Sigma x_i J_i(\xi) = 0$ which correspond to the tangent cones from the points (x) of space to the surface are of order 9. The point P_0 is fivefold and the points $P_1, P_2, \ldots, P_6$ are double points on these curves. Since the surface is of class 20 (Art. 20) there are twelve other points common to these curves. These twelve points correspond to pinch-points on the double curve. The tangent cone from a generic point to the surface is of genus 12.

22. The parabolic curve is of order 32. It touches each right line on the surface twice, intersects each generating conic in eight points and each directrix conic in twelve points. It has a double point at each pinch-point of the double curve. Its genus is 57. The developable of the stationary planes is of class 48 and order 80. These planes are determined by imposing the condition that (14) has a triple root.

23. The double developable is of class 111. It has as components the twelve pencils of planes (Art. 4) which contain a right line and a quintic of genus 1, and the cubic developable (Art. 2) of the planes which intersect the surface in a generating conic and a quartic of genus 1. The section by a generic plane of the residual developable is a non-composite rational sextic.

24. There are two hundred and thirty-eight triple tangent planes. The section of the surface by such a plane is composite since it has eleven double points. There are: (a) six planes which contain two lines and a quartic of genus 1; (b) one hundred and forty-four which contain a line and a rational quartic; (c) twenty-four which contain a generating conic and a rational quartic; (d) thirty-two which contain a directrix conic and a rational quartic; (e) thirty-two which contain two rational cubics. The two cubics in a plane of

type (e) belong to two systems of cubics on the surface (Art. 17) such that the curves of one system intersect those of the other in three points. In each pencil of cubics on the surface there are two cubics which lie in a plane.

25. The double curve intersects the generating conics in six points, the directrix conics in five points, and the right lines on the surface in three points, (Arts. 9 and 16). Hence, the curve $\sigma=0$ corresponding to the double curve has a fivefold point at P_0, threefold points at $P_1, P_2, \ldots, P_6$ and is of order 11. Each curve $\Sigma u_i f_i=0$ which passes through a given point of $\sigma=0$ passes through a second fixed point. To each triple point of the surface correspond three points of σ such that each curve $\Sigma u_i f_i=0$ through one of these points passes through the other two. These points are double points on $\sigma=0$. The curve $\sigma=0$ is of genus 11. Its points determine, by means of equations (16), the tangent planes to the surface along the double curve. The developable of these planes is of class 26.

26. The quartic curves in the ξ-plane which have a double point at P_0 and pass through $P_1, P_2, \ldots, P_6$ form a linear system ∞^5. If $f_i(\xi_1, \xi_2, \xi_3)=0$ $(i=1, 2, \ldots, 6)$ are six linearly independent curves of this system, the sextic surface

$$x_i=f_i(\xi_1, \xi_2, \xi_3) \qquad i=1, 2, \ldots, 6 \tag{17}$$

defined by them belongs to a space of five dimensions S_5 and has the surface (10) as its projection from a generic line on an S_3. A generic hyperplane section of the surface is of genus 2, since it is in (1, 1) correspondence with a quartic curve of genus 2 in the ξ-plane.

27. No sextic surface whose hyperplane sections are of genus 2 can belong to a space of more than five dimensions. For, the sextic curve of section by a generic hyperplane belongs to an S_4 (at most). The S_5 determined by such an S_4 and a generic point on the surface contains the surface. If such a surface belongs to an S_5 (*i. e.*, does not lie in an S_4), it is either a cone projecting from a point a sextic curve of genus 2, or a rational surface of the type of equations (17). For, suppose first that the surface is ruled. Let S_4 be a four-space which contains two of its generators. If the generators of the surface intersect S_4 in a fixed point, the surface is a cone. If not, the locus of the intersection of the generators with S_4 is a plane quartic curve of genus 2. Hence, such a surface can be defined in infinitely many ways by a (1, 1) correspondence between two plane quartics. But the ruled surface defined by such a correspondence is of order 8 unless the quartics intersect. If the quartics intersect, their planes define a space of (at most) four dimensions which contains the ruled surface. If the surface is unruled, its projection from a

generic line on an S_3 is an unruled sextic surface with generic plane section of genus 2. The given surface is therefore rational (Art. 2) and its equations can be birationally reduced to the form (17).

28. To the cubic curves in the ξ-plane which pass through $P_0, P_1, \ldots, P_6$ correspond on the surface (17) a linear system ∞^2 of quartic curves (cf. Art. 18). Each of these quartics defines an S_3 in which it lies. Since through any two given points of the surface a quartic of the system passes, each bisecant line to the surface lies in such an S_3. In a generic bundle of quartics of the system in the ξ-plane which corresponds to the hyperplane sections of the surface, there is a unique pencil which breaks up into a fixed cubic through $P_0, P_1, \ldots, P_6$ and a pencil of lines. Hence, a generic point of S_5 lies in the S_3 of a single quartic of genus 1 on the surface. The apparent double points of the surface from the given point are the two apparent double points of the quartic lying in the S_3 which passes through the given point.

29. Through a generic point P on the surface, there pass a pencil of quartic curves of genus 1 on the surface. These quartic curves have a second fixed point P' in common determined by the ninth intersection of the corresponding cubics in the ξ-plane. The line PP' is thus common to the S_3 defined by all these quartics. It intersects the plane of each of the generating conics on the surface. For, let C_2 be such a conic. Then C_2 and the pencil of quartics through P are the sections of the surface by a pencil of hyperplanes, since the corresponding curves in the ξ-plane constitute a pencil of quartics with a double point at P_0 and passing through $P_1, P_2, \ldots, P_6$ (Art. 26). The basis S_3 of this pencil of S_4 contains the plane of the conic and the line PP'. Hence the line and the plane intersect. The system ∞^2 of lines PP' joining all the pairs of corresponding points of the given surface thus constitute a rectilinear generation of the cubic three-spread V_3^3 defined by the planes of the generating conics of the surface.*

30. The given surface (17) lies on a web of hyperquadrics in S_5. For, the hyperquadrics which pass through eight generic fixed points of one quartic of the system of Art. 28, through six such points of another and through three generic fixed points of a directrix conic, contain these curves. They, therefore, contain the surface, since they have five points in common with each generating

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* The lines joining corresponding pairs of points on the surface (11) in S_3 form a (3, 1) congruence. For, these lines are the projections from a fixed generic line l, of the lines on V_3^3 in S_6. Three such lines in S_6 intersect a generic plane through l since V_3^3 is of order 3, and one lies in a generic S_4 through l since the section of V_3^3 by S_4 is a ruled cubic with a single rectilinear directrix. Hence, three lines of the congruence by S_3 pass through a given point and one lies in a given plane.

conic. Since we have imposed, at most, seventeen conditions on the twenty-one homogeneous parameters in the equation of an hyperquadric in S_5, there exists a web of such hypersurfaces which contain the given surface.

The hyperquadrics of this web do not all contain V_3^3. For, the hyperquadrics which contain three skew planes of V_3^3, and hence contain the three-spread, constitute a bundle.* The given surface is thus the complete intersection of V_3^3 with an hyperquadric. Conversely, the intersection of a given hyperquadric in S_6 by a given cubic three-spread generated by planes is, in general, a surface of the type of equations (17), since the projection of the intersection from a generic line onto a generic S_3 is a surface of order 6 generated by conics whose planes envelope a rational developable of class 3.

31. Let the points of a non-singular hyperquadric $F_2=0$ be put in (1, 1) correspondence with the lines of S_3. To the section of $F_2=0$ by V_3^3 corresponds a (3, 3) congruence, since each plane contained in $F_2=0$ intersects V_3^3 in three points.

Conversely, let there be given a (3, 3) congruence which does not belong to a linear complex. The section of the congruence by a generic linear complex is a ruled sextic which does not belong to a linear congruence and hence is of genus not greater than 2. If the genus of such a section is equal to 2, the complex can not be generated by pencils of lines, since all the generating pencils would have a fixed line in common (Art. 27), and the congruence would belong to a special linear complex. The congruence is therefore rational and is generated in one, and only one, way by reguli. It belongs to a bundle of quadratic complexes, any two of which have for residual intersection a linear congruence which has in common with the given congruence a ruled quartic surface of genus 1 (Art. 30, footnote). A generic line in space belongs to one and only one such residual linear congruence of intersection of two quadratic complexes of the bundle (Art. 28).

32. To find the equation of the system of quadrics determined by the reguli which generate the congruence, let $p_{ij}(i=1, 2, 3, 4)$ be the current co-

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*The parametric equations of V_3^3, referred to a suitable coordinate system, may be written in the form

$$x_1=k_1l_1,\quad x_2=k_1l_2,\quad x_3=k_1l_3,\quad x_4=k_2l_1,\quad x_5=k_2l_2,\quad x_6=k_2l_3,$$

where k_i and l_i are parameters. The hyperquadrics which contain V_3^3 belong to the bundle

$$\lambda_1(x_1x_5-x_2x_4)+\lambda_2(x_1x_6-x_3x_4)+\lambda_3(x_2x_6-x_3x_5)=0.$$

Each hyperquadric of this bundle has a double line. The intersection of any two hyperquadrics of the bundle consists of V_3^3 and an S_3 which intersects the surface (17) in a quartic curve of genus 1.

ordinates in S_5 and let the equation of $F_2=0$ be $p_{12}p_{34}+p_{13}p_{42}+p_{14}p_{23}=0$ so that the coordinates of a point on $F_2=0$ are also the coordinates

$$p_{ij}=y_ix_j-x_iy_j \qquad i,\, j=1,\, 2,\, 3,\, 4,$$

of a line in S_3.

The equations of a generating plane of V_3^3 can be written in the form (cf. Art. 30, footnote),

$$\Sigma\phi_{ij}^{(1)}p_{ij}=0, \quad \Sigma\phi_{ij}^{(2)}p_{ij}=0, \quad \Sigma\phi_{ij}^{(3)}p_{ij}=0, \tag{18}$$

where $\phi_{ij}^{(h)}\equiv-\phi_{ji}^{(h)}\equiv a_{ij}^{(h)}k_1+b_{ij}^{(h)}k_2$, $i, j=1, 2, 3, 4$, $h=1, 2, 3$, $a_{ij}^{(h)}$ and $b_{ij}^{(h)}$ are constants and k_1, k_2 are parameters which define the planes of V_3^3.

If we put $p_{ij}=y_ix_j-y_jx_i$, equations (18) take the form

$$\Sigma\phi_{ij}^{(1)}y_ix_j=0, \quad \Sigma\phi_{ij}^{(2)}y_ix_j=0, \quad \Sigma\phi_{ij}^{(3)}y_ix_j=0.$$

The condition that these equations have solutions in (y) other than

$$\frac{y_1}{x_1}=\frac{y_2}{x_2}=\frac{y_3}{x_3}=\frac{y_4}{x_4}$$

is that all the third order determinants in the matrix

$$\left\|\begin{matrix} \phi_{12}^{(1)}x_2+\phi_{13}^{(1)}x_3+\phi_{14}^{(1)}x_4 & \phi_{21}^{(1)}x_1+\phi_{23}^{(1)}x_3+\phi_{24}^{(1)}x_4 & \phi_{31}^{(1)}x_1+\phi_{32}^{(1)}x_2+\phi_{34}^{(1)}x_4 & \phi_{41}^{(1)}x_1+\phi_{42}^{(1)}x_2+\phi_{43}^{(1)}x_3 \\ \phi_{12}^{(2)}x_2+\phi_{13}^{(2)}x_3+\phi_{14}^{(2)}x_4 & \phi_{21}^{(2)}x_1+\phi_{23}^{(2)}x_3+\phi_{24}^{(2)}x_4 & \phi_{31}^{(2)}x_1+\phi_{32}^{(2)}x_2+\phi_{34}^{(2)}x_4 & \phi_{41}^{(2)}x_1+\phi_{42}^{(2)}x_2+\phi_{43}^{(2)}x_3 \\ \phi_{12}^{(3)}x_2+\phi_{13}^{(3)}x_3+\phi_{14}^{(3)}x_4 & \phi_{21}^{(3)}x_1+\phi_{23}^{(3)}x_3+\phi_{24}^{(3)}x_4 & \phi_{31}^{(3)}x_1+\phi_{32}^{(3)}x_2+\phi_{34}^{(3)}x_4 & \phi_{41}^{(3)}x_1+\phi_{42}^{(3)}x_2+\phi_{43}^{(3)}x_3 \end{matrix}\right\|$$

vanish. These determinants have in common the factor $\Delta\equiv\Sigma\Phi_{ij}x_i$, x_j where

$$\Phi_{11}\equiv\begin{vmatrix} \phi_{14}^{(1)}\phi_{21}^{(1)}\phi_{31}^{(1)} \\ \phi_{14}^{(2)}\phi_{21}^{(2)}\phi_{31}^{(2)} \\ \phi_{14}^{(3)}\phi_{21}^{(3)}\phi_{31}^{(3)} \end{vmatrix}, \quad 2\Phi_{12}\equiv\begin{vmatrix} \phi_{12}^{(1)}\phi_{24}^{(1)}\phi_{31}^{(1)} \\ \phi_{12}^{(2)}\phi_{24}^{(2)}\phi_{31}^{(2)} \\ \phi_{12}^{(3)}\phi_{24}^{(3)}\phi_{31}^{(3)} \end{vmatrix}+\begin{vmatrix} \phi_{12}^{(1)}\phi_{14}^{(1)}\phi_{32}^{(1)} \\ \phi_{12}^{(2)}\phi_{14}^{(2)}\phi_{32}^{(2)} \\ \phi_{12}^{(3)}\phi_{14}^{(3)}\phi_{32}^{(3)} \end{vmatrix},$$

etc. Hence, $\Delta=0$ is the equation required. Six of these surfaces break up into pairs of planes.

The opposite reguli on this system of surfaces generate another congruence of the same type. It corresponds to the section of the hyperquadric $p_{12}p_{34}+p_{13}p_{42}+p_{14}p_{23}=0$ by the V_3^3

$$p_{mn}=a_{ij}^{(1)}k_1l_1+a_{ij}^{(2)}k_1l_2+a_{ij}^{(3)}k_1l_3+b_{ij}^{(1)}k_2l_1+b_{ij}^{(2)}k_2l_2+b_{ij}^{(3)}k_2l_3,$$

where i, j, m, n are the four numbers 1, 2, 3, 4 in some order and k_1, k_2, l_1, l_2, l_3 are parameters.

33. If the surface (17) in S_5 has a double point, the curves in the ξ-plane which correspond to the sections of the surface by the hyperplanes through the double point constitute a web of curves of genus 1. If these curves are proper quartics with a double point additional to P_0, two of the fundamental points $P_1, P_2, \ldots, P_6$ are consecutive at this point; if they degenerate into a web of cubics and a fixed line through P_0, two simple fundamental points lie on the fixed line. Conversely, if two simple fundamental points are consecutive, or are collinear with* P_0, the surface (17) has a double point. Sixteen of the directrix conics pass through the double point. These conics are consecutive in pairs. The projection of the surface on S_3 has a double point distinct from the double curve.

Not more than six distinct double points can exist on the surface (17). Six such points exist when $P_1, P_2, \ldots, P_6$, are consecutive in pairs, in such a way that consecutive points are collinear with P_0. The six double points lie in pairs on three generating conics, each of which reduces to a right line counted twice. Three generating conics touch a generic hyperplane (cf. Art. 20).

34. If three simple fundamental points are collinear the surface has a directrix line. Six directrix conics degenerate into this line and a component of a generating conic. Such a surface arises as the intersection of a V_3^3 with an hyperquadric which contains a rectilinear generator of V_3^3 (Art. 30). The residual section of the surface by an hyperplane which contains the directrix line is a rational quintic which intersects the directrix line in three points.

The surface can have at most four directrix right lines without having a double point. The surface has four such lines when the points $P_1, P_2, \ldots, P_6$ are the vertices of a complete quadrilateral. The four directrix lines do not intersect. There are eight non-composite directrix conics on the surface. None of these intersects the directrix lines.

35. If the fundamental points $P_1, P_2, \ldots, P_6$ lie on a conic C', then to C' corresponds a conic C which intersects each generating conic in two points. Any two generating conics form, with C, the section of the surface by an hyperplane. Hence, any two planes of V_3^3 lie in an S_4 and intersect so that V_3^3 is a three-spread cone which projects a ruled cubic surface (belonging to an S_4) from a fixed point. Conversely, if V_3^3 is conical, any two generating conics of the given surface lie in an S_4 whose residual intersection with the surface is a conic which intersects all the generating conics in two points so that $P_1, P_2, \ldots, P_6$ lie on a conic. None of the directrix conics intersects C.

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* These two configurations are birationally equivalent.

If C' also passes through P_0, all the generating conics pass through the triple point of V_3^3 which is, in this case, a triple point on the given surface. Conversely, if the surface (17) has a triple point, all the generating conics pass through the triple point, and this point is triple on V_3^3. The S_4 determined by the planes of any two generating conics contains, as residual intersection with the surface, a directrix conic which passes through the triple point and intersects each generating conic on the surface in a second point. By a birational transformation in the ξ-plane, the conic C' may be transformed into a right joining four simple fundamental points.

The given surface can have two triple points. This happens when P_1 and P_2 are each consecutive to P_0 and $P_3P_6P_5P_6$ are collinear. The planes of V_3^3 join the points of a cubic curve to a fixed right line l which is triple on l. The triple points of the surface are the points of intersection of l with an hyperquadric on which the surface lies (Art. 30). They may be consecutive.

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